Consistency of Kaluza-Klein Sphere Reductions of Symmetric Potentials

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ABSTRACT

In a recent paper, the complete (non-linear) Kaluza-Klein Ansatz for the consistent embedding of certain scalar plus gravity subsectors of gauged maximal supergravity in $D = 4$, 5 and 7 was presented, in terms of sphere reductions from $D = 11$ or type IIB supergravity. The scalar fields included in the truncations were the diagonal fields in the $SL(N, R)/SO(N)$ scalar submanifolds of the full scalar sectors of the corresponding maximal supergravities, with $N = 8$, 6 and 5. The embeddings were used for obtaining an interpretation of extremal $D = 4$, 5 or 7 AdS domain walls in terms of distributed M-branes or D-branes in the higher dimensions. Although strong supporting evidence for the correctness of the embedding Ansätze was presented, a full proof of the consistency was not given. Here, we complete the proof, by showing explicitly that the full set of higher-dimensional equations of motion are satisfied if and only if the lower-dimensional fields satisfy the relevant scalar plus gravity equations.

1 Introduction

One of the more intriguing outcomes of recent work on the AdS/CFT correspondence has been a renewed effort to understand how the lower-dimensional gauged supergravities arise as Kaluza-Klein sphere reductions from $D = 11$ or type IIB supergravity. It was long ago demonstrated how the reductions work at the linearised level, but few complete non-linear results existed. A proof of the consistency of the S^7 reduction from $D = 11$ was presented, although the Kaluza-Klein Ansatz for the field-strength sector was not fully explicit [1]. It was generally assumed that the other cases, namely the S^4 reduction of $D = 11$, and the S^5 reduction of type IIB, would be consistent too, but until recently no results for these cases had been obtained. In recent work, a fully explicit reduction Ansatz for the $SO(5)$-gauged $N = 4$ $D = 7$ case has been obtained [2, 3].[1] Explicit results have also been obtained for various truncations of the full maximal supergravities. These include truncations to the maximal abelian subgroups $U(1)^4$, $U(1)^3$ and $U(1)^2$ in $D = 4$, 5 and 7 [5]; the truncation to $SU(2)$-gauged $N = 2$ in $D = 7$ [6]; to $SU(2) \times U(1)$ gauged $N = 4$ in $D = 5$ [7]; and to $SO(4)$ gauged $N = 4$ in $D = 4$ [8]. For many purposes, if the fields that participate in the lower-dimensional solutions of interest lie within these truncated subsectors, the truncated reduction is much easier to use, since it is usually much simpler than the full maximal result.[2]

Another truncation that allows for relatively simple although still non-trivial sphere reductions is where one retains only the metric and a certain subset of the scalar fields of the lower-dimensional gauged supergravity; one can keep just certain scalars contained in the $SL(N, R)/SO(N)$ subset of the full scalar coset manifold, which can be described by a symmetric tensor T_{ij}. In $D = 4$, 5 and 7 these subsets correspond to $N = 8$, 6 and 5 respectively. Specifically, one can consistently truncate to the diagonal scalars, $T_{ij} = X_i \, \delta_{ij}$, where $\prod_i X_i = 1$. Thus there are 7, 5 and 4 independent scalars in the $D = 4$, 5 and 7 cases. As shown in [9], where the reduction Ansätze for these scalar subsectors were presented,

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[1]The complete bosonic reduction Ansatz for another case, namely the local S^4 reduction of massive type IIA supergravity to $SU(2)$ gauged $N = 2$ supergravity in $D = 6$ has also been obtained [4].

[2]We emphasise that in this discussion we are considering only the "remarkable" Kaluza-Klein sphere reductions, where there is no group-theoretic understanding for why the consistency is achievable. In particular, some of the scalar fields parameterise inhomogeneous distortions of the sphere. These contrast with, for example, torus reductions, where the consistency of the truncation to the massless sector is guaranteed by group theory.

one can actually discuss all cases where the lower dimension D is related to N by

$$N = \frac{4(D-2)}{D-3} \, , \tag{1}$$

corresponding to supersymmetric higher-dimensional theories, in a uniform way. The only integer possibilities are $(D, N) = (4, 8)$, $(5, 6)$ and $(7, 5)$, as listed above. (Some proposals for other scalar truncations were presented recently in [10].) In [9], extremal AdS domain wall solutions in these dimensions were derived, with the general set of $(N-1)$ independent charge parameters. By using the reduction Ansätze to oxidise the solutions to the higher dimensions, it was shown how they can be interpreted as continuous distributions of M-branes or D-branes [9]. (Various special cases were obtained also in [11, 12, 13, 14, 15].)

Certain consistency checks for the reduction Ansätze presented in [9] were conducted there, but a full demonstration of the consistency was not given. Here, we complete the argument by checking all the higher-dimensional equations of motion, and verifying that indeed they are satisfied by the Ansätze of [9], if and only if the lower-dimensional equations of motion are satisfied.[3] Of course these calculations would be subsumed by complete demonstrations of the consistency of the maximal supergravity reductions in $D = 4$, 5 and 7. Such a complete proof exists for $D = 7$ [2, 3], and implicitly for $D = 4$ [1], but not yet for $D = 5$. Thus the results presented here provide new and independent evidence for the conjectured consistency in all the cases.

2 The Scalar Theories, and the Reduction Ansätze

The truncated lower-dimensional gravity plus scalar theory is described by the following Lagrangian in D dimensions [9]:

$$e^{-1} \mathcal{L}_D = R - \tfrac{1}{2}(\partial \vec{\varphi})^2 - V \, , \tag{2}$$

where the potential V is given by

$$V = -\tfrac{1}{2} g^2 \left((\sum_{i=1}^{N} X_i)^2 - 2 \sum_{i=1}^{N} X_i^2 \right) . \tag{3}$$

(In $D = 4$, 5 and 7, we shall have $N = 8$, 6 and 5 respectively.) The N quantities X_i are parameterised in terms of $(N-1)$ independent dilatonic scalars $\vec{\varphi}$ as follows:

$$X_i = e^{-\frac{1}{2} \vec{b}_i \cdot \vec{\varphi}} \, , \tag{4}$$

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[3]Note that substituting into the higher-dimensional Lagrangian and integrating out the sphere directions could never, *per se*, yield a proof of consistency.

where the $\vec{b}_i$ satisfy

$$\vec{b}_i \cdot \vec{b}_j = 8\delta_{ij} - \frac{8}{N}\,, \qquad \sum_i \vec{b}_i = 0\,, \qquad (\vec{u} \cdot \vec{b}_i)\,\vec{b}_i = 8\vec{u}\,, \tag{5}$$

The middle equation here expresses the fact that the N quantities X_i are subject to the condition

$$\prod_{i=1}^{N} X_i = 1\,. \tag{6}$$

The last equation in (5) allows us to express the dilatons $\vec{\varphi}$ in terms of the X_i:

$$\vec{\varphi} = -\tfrac{1}{4} \sum_i \vec{b}_i \log X_i\,. \tag{7}$$

The equations of motion for the scalar fields, following from (2), are

$$\Box\vec{\varphi} = \frac{\partial V}{\partial \vec{\varphi}}\,. \tag{8}$$

From (4) it follows that $\partial X_i/\partial\vec{\varphi} = -\tfrac{1}{2}\vec{b}_i\, X_i$, and hence the equations of motion (8) become

$$\Box\vec{\varphi} = \tfrac{1}{2}g^2 \sum_i \vec{b}_i \Big(X_i \sum_j X_j - 2X_i^2 \Big)\,. \tag{9}$$

Note that we can also write the scalar equations of motion as

$$\Box \log X_i = 2g^2 \Big(2X_i^2 - X_i \sum_j X_j - \tfrac{2}{N} \sum_k X_k^2 + \tfrac{1}{N}(\sum_j X_j)^2 \Big)\,. \tag{10}$$

The Einstein equation following from (2) is

$$R_{\mu\nu} = \tfrac{1}{4}X_i^{-2}\, \partial_\mu X_i\, \partial_\nu X_i + \frac{1}{D-2}\, V\, g_{\mu\nu}\,. \tag{11}$$

The Kaluza-Klein sphere reduction Ansätze for obtaining these theories from the higher dimension were presented in [9], and are as follows:

$$
\begin{aligned}
d\hat{s}^2 &= \Delta^{\frac{2}{D-1}}\, ds_D^2 + \frac{1}{g^2}\, \Delta^{-\frac{D-3}{D-1}} \sum_i X_i^{-1}\, d\mu_i^2\,, \\
\hat{F} &= g \sum_i (2X_i^2\,\mu_i^2 - \Delta\, X_i)\,\epsilon_{(D)} + \frac{1}{2g} \sum_i X_i^{-1} *dX_i \wedge d(\mu_i^2)\,,
\end{aligned} \tag{12}
$$

where

$$\Delta = \sum_i X_i\,\mu_i^2\,, \tag{13}$$

and the μ_i are a set of N "direction cosines" that satisfy the constraint

$$\sum_i \mu_i^2 = 1\,. \tag{14}$$

In (12), $\epsilon_{(D)}$ denotes the volume form of the D-dimensional metric ds_D^2. Note that if all the scalars X_i are trivial ($X_i = 1$), the internal part of the metric becomes $\sum_i d\mu_i^2$, which is the metric on the unit $(N-1)$-sphere.

The D-form field strength $\hat{F}$ in (12) is the 4-form of eleven-dimensional supergravity for the case $D = 4$, the Hodge dual of this 4-form for the case $D = 7$, and it is the self-dual 5-form of the type IIB theory when $D = 5$. Note that in each case, given the nature of the Ansatz, the relevant Bianchi identity and field equation for $\hat{F}$ are simply

$$d\hat{F} = 0 , \qquad d\hat{*}\hat{F} = 0 . \qquad (15)$$

3 The Consistency of the Reduction

It was shown in [9] that the D-form field-strength Ansatz in (12) satisfies the Bianchi identity $d\hat{F} = 0$, provided that the scalar fields X_i satisfy precisely the lower-dimensional equations of motion (10). This calculation is a straightforward one, and we shall not repeat it here. It is harder to show that $\hat{F}$ satisfies the field equation $d\hat{*}\hat{F} = 0$, because this involves taking a Hodge dual of the field strength $\hat{F}$. This is what we shall now address.

3.1 The Field Equation for $\hat{F}$

The complication here is that the $(N-1)$-sphere is being coordinatised by N quantities μ_i subject to the constraint (14). It seems that the best way to proceed is to eliminate one of the μ_i in favour of the others, using (14). To that end, we split the μ_i as $\mu_i = (\mu_\alpha, \mu_0)$, and solve for μ_0 in terms of the μ_α.

If we consider first the metric

$$ds^2 = \sum_i X_i^{-1} d\mu_i^2 , \qquad (16)$$

then in terms of the μ_α we can write it as $ds^2 = g_{\alpha\beta} d\mu_\alpha d\mu_\beta$, where

$$g_{\alpha\beta} = X_\alpha^{-1} \delta_{\alpha\beta} + \frac{1}{X_0 \mu_0^2} \mu_\alpha \mu_\beta . \qquad (17)$$

(We have, of course, used the identity

$$d\mu_0 = -\frac{\mu_\alpha}{\mu_0} d\mu_\alpha , \qquad (18)$$

which follows from (14).)

It is straightforward to invert the metric $g_{\alpha\beta}$ given in (17). The result is

$$g^{\alpha\beta} = X_\alpha \delta_{\alpha\beta} - \Delta^{-1} \mu_\alpha \mu_\beta X_\alpha X_\beta . \qquad (19)$$

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It is also easy to establish that

$$\det(g_{\alpha\beta}) = \frac{\Delta}{\mu_0^2}\,. \tag{20}$$

Note that it follows from the metric Ansatz in (12) that the determinant of the higher-dimensional metric $d\hat{s}^2$ is given by

$$\det(\hat{g}) = \frac{\Delta^{\frac{4}{D-1}}}{g^{2N-2}\,\mu_0^2}\,\det(g_D)\,, \tag{21}$$

where g_D denotes the D-dimensional spacetime metric ds_D^2 and g in the denominator is just the gauge coupling constant (not to be confused with the determinant of the higher-dimensional metric $\hat{g}$ or the one for the lower dimension, g_D.)

Now let us look at the field-strength Ansatz. We shall use the convention that $\varepsilon_{M_1\cdots M_D}$ always means the tensor density, which is the pure numbers $\pm 1, 0$. So the Ansatz for $\hat{F}$ in (12) is

$$\begin{aligned}
\hat{F}_{\nu_1\cdots\nu_D} &= gU\sqrt{-g_D}\,\varepsilon_{\nu_1\cdots\nu_D}\,, \\
\hat{F}_{\nu_1\cdots\nu_{D-1}\alpha} &= \frac{1}{g}\sqrt{-g_D}\,\varepsilon_{\nu_1\cdots\nu_{D-1}\rho}\,g_D^{\rho\sigma}\,(X_\alpha^{-1}\,\partial_\sigma X_\alpha - X_0^{-1}\,\partial_\sigma X_0)\,\mu_\alpha\,,
\end{aligned} \tag{22}$$

where

$$U \equiv \sum_i (2X_i^2\,\mu_i^2 - \Delta\,X_i)\,, \tag{23}$$

and g_D denotes the D-dimensional spacetime metric ds_D^2.

We can now calculate the upper-index components of $\hat{F}$. In fact, what we really need is these components multiplied by $\sqrt{\hat{g}}$. From the results above we find

$$\begin{aligned}
\sqrt{-\hat{g}}\,\hat{F}^{\nu_1\cdots\nu_D} &= \frac{U}{g^{N-2}\mu_0\,\Delta^2}\,\varepsilon^{\nu_1\cdots\nu_D}\,, \\
\sqrt{-\hat{g}}\,\hat{F}^{\nu_1\cdots\nu_{D-1}\alpha} &= \frac{1}{g^{N-2}\mu_0}\,\varepsilon^{\nu_1\cdots\nu_{D-1}\sigma}\,\partial_\sigma\Big(\frac{X_\alpha\,\mu_\alpha}{\Delta}\Big)\,.
\end{aligned} \tag{24}$$

($\varepsilon^{M_1\cdots M_D}$ is the tensor density that takes the values 0, ± 1, and is numerically equal to $\varepsilon_{M_1\cdots M_D}$.)

One can directly verify from these expressions that the field equation is satisfied, namely that

$$\partial_M\Big(\sqrt{-\hat{g}}\,\hat{F}^{N_1\cdots N_{D-1}M}\Big) = 0\,. \tag{25}$$

However, it is more elegant to do this by using (24) first to construct the Hodge dual of $\hat{F}$ itself. To do this, we make the following definitions:

$$P \equiv \frac{1}{n!}\,\varepsilon_{\alpha_1\cdots\alpha_n}\,d\mu_{\alpha_1}\cdots d\mu_{\alpha_n}\,,$$

$$Q_\alpha \equiv \frac{1}{(n-1)!}\,\varepsilon_{\alpha\beta_1\cdots\beta_{n-1}}\,d\mu_{\beta_1}\cdots d\mu_{b_{n-1}}\,,$$

$$W \equiv \frac{1}{n!}\,\varepsilon_{ij_1\cdots j_n}\,\mu_i\,d\mu_{j_1}\cdots d\mu_{j_n}\,,$$

$$Z_i \equiv \frac{1}{(n-1)!}\,\varepsilon_{ijk_1\cdots k_{n-1}}\,\mu_j\,d\mu_{k_1}\cdots d\mu_{k_{n-1}}\,, \tag{26}$$

where $n = N-1$. Note that what we have done here is to define P and Q_α with respect to the reduced set of $n = N-1$ coordinates μ_α, while W and Z_i are defined with respect to the full set of N coordinates μ_i. (Some analogous formulae and manipulations are presented also in [3].)

Now, we can establish the following:

$$W = \frac{1}{\mu_0}\,P\,,$$

$$Z_0 = \mu_\beta\,Q_\beta\,, \qquad Z_\alpha = \frac{1}{\mu_0}\,(-Q_\alpha + \mu_\alpha\,\mu_\beta\,Q_\beta)\,,$$

$$d\mu_\alpha \wedge Q_\beta = P\,\delta_{\alpha\beta}\,,$$

$$d\mu_i \wedge Z_j = -(\delta_{ij} - \mu_i\,\mu_j)\,W\,,$$

$$dQ_\alpha = 0\,, \qquad dW = 0\,, \qquad dZ_i = n\,\mu_i\,W\,. \tag{27}$$

From (24), it is evident that we have

$$\hat{*}\hat{F}_{\alpha_1\cdots\alpha_n} = \frac{U}{g^{N-2}\mu_0\,\Delta^2}\,\varepsilon_{\alpha_1\cdots\alpha_n}\,, \qquad \hat{*}\hat{F}_{\alpha_1\cdots\alpha_{n-1}\nu} = -\frac{1}{g^{N-2}\mu_0}\,\varepsilon_{\alpha_1\cdots\alpha_{n-1}\beta}\,\partial_\nu\Big(\frac{X_\beta\,\mu_\beta}{\Delta}\Big)\,. \tag{28}$$

Note that here, and in many other formulae, we are using a "generalised Einstein summation convention," in which any dummy index that appears two or more times in an expression is understood to be summed over. It will always be clear from context whether an index is a dummy or not.

After some algebra, we can show from the above definitions and properties that this can be written as

$$\hat{*}\hat{F} = \frac{1}{g^{N-2}}\Big(\frac{U}{\Delta^2}\,W + \partial_\nu(\frac{X_i\,\mu_i}{\Delta})\,dx^\nu \wedge Z_i\Big)\,. \tag{29}$$

To check that the equation of motion $d\hat{*}\hat{F} = 0$ is satisfied, we just have to make use of the various lemmata established above. Thus we have

$$
\begin{aligned}
g^{N-2}d\hat{*}\hat{F} &= \partial_\nu\Big(\frac{U}{\Delta^2}\Big)\,dx^\nu \wedge W - \partial_\nu\Big(\frac{X_i\,\mu_i}{\Delta}\Big)\,dx^\nu \wedge dZ_i - \partial_{\mu_j}\partial_\nu\Big(\frac{X_i\,\mu_i}{\Delta}\Big)\,dx^\nu \wedge d\mu_j \wedge Z_i\,, \\
&= \partial_\nu\Big(\frac{U}{\Delta^2}\Big)\,dx^\nu \wedge W - n\,\mu_i\,\partial_\nu\Big(\frac{X_i\,\mu_i}{\Delta}\Big)\,dx^\nu \wedge W \\
&\quad + \partial_{\mu_j}\partial_\nu\Big(\frac{X_i\,\mu_i}{\Delta}\Big)\,dx^\nu \wedge W\,(\delta_{ij} - \mu_i\,\mu_j)\,, \\
&= \partial_\nu\Big(\frac{U}{\Delta^2}\Big)\,dx^\nu \wedge W + \partial_\nu\Big(\delta_{ij}\,\frac{X_i}{\Delta} - \frac{2X_i\,X_j\,\mu_i\,\mu_j}{\Delta^2}\Big)\,dx^\nu \wedge W\,(\delta_{ij} - \mu_i\,\mu_j)\,,
\end{aligned}
$$

$$= \partial_\nu \left(\frac{U}{\Delta^2} \right) dx^\nu \wedge W - \partial_\nu \left(\frac{U}{\Delta^2} \right) dx^\nu \wedge W = 0 \,. \tag{30}$$

Note that in various steps above, we have made use of the fact that the μ_i can be taken freely inside the ∂_ν derivative, and that therefore, for instance, a term like $\mu_i \, \partial_\nu (X_i \, \mu_i / \Delta)$ is equal to $\partial_\nu (X_i \, \mu_i^2 / \Delta)$, which is therefore zero since $X_i \, \mu_i^2 = \Delta$. This completes the checking of the consistency of the higher-dimensional field equation for $\hat{F}$.

3.2 The Einstein Equation

3.2.1 Calculation of the Ricci Tensor

To check the various components of the higher-dimensional Einstein equation, we first calculate the curvature tensor for the metric Ansatz. From now on, since no generality is lost, we set the gauge coupling g equal to 1 for simplicity. The metric can be written as

$$d\hat{s}^2 = \Delta^a \, ds_D^2 + \Delta^{-b} \sum_i X_i^{-1} \, d\mu_i^2 \,, \tag{31}$$

where

$$a = \frac{2}{D-1} \,, \qquad b = \frac{D-3}{D-1} \,. \tag{32}$$

From this, we find that the affine connection $\hat{\Gamma}^M{}_{NP} = \frac{1}{2} \hat{g}^{MQ} \left(\partial_N \, g_{QP} + \partial_P \, g_{QN} - \partial_Q \, g_{NP} \right)$ is given by

$$
\begin{aligned}
\hat{\Gamma}^\mu{}_{\nu\rho} &= \Gamma^\mu{}_{\nu\rho} + \tfrac{1}{2} a \, \Delta^{-1} \left(\delta^\mu_\rho \, \partial_\nu \, \Delta + \delta^\mu_\nu \, \partial_\rho \, \Delta - g_{\nu\rho} \, \partial^\mu \, \Delta \right), \\
\hat{\Gamma}^\mu{}_{\nu\alpha} &= \tfrac{1}{2} a \, \Delta^{-1} \, \delta^\mu_\nu \, \partial_\alpha \, \Delta \,, \\
\hat{\Gamma}^\alpha{}_{\mu\nu} &= -\tfrac{1}{2} a \, g_{\mu\nu} \, \partial^\alpha \, \Delta \,, \\
\hat{\Gamma}^\alpha{}_{\beta\mu} &= -\tfrac{1}{2} b \, \Delta^{-1} \, \delta^\alpha_\beta \, \partial_\mu \, \Delta + \tfrac{1}{2} g^{\alpha\gamma} \, \partial_\mu \, g_{\beta\gamma} \,, \\
\hat{\Gamma}^\mu{}_{\alpha\beta} &= \tfrac{1}{2} b \, g_{\alpha\beta} \, \Delta^{-2} \, \partial^\mu \, \Delta - \tfrac{1}{2} \Delta^{-1} \, \partial^\mu \, g_{\alpha\beta} \,, \\
\hat{\Gamma}^\alpha{}_{\beta\gamma} &= \Gamma^\alpha{}_{\beta\gamma} - \tfrac{1}{2} b \, \Delta^{-1} \left(\delta^\alpha_\gamma \, \partial_\beta \, \Delta + \delta^\alpha_\beta \, \partial_\gamma \, \Delta - g_{\beta\gamma} \, \partial^\alpha \, \Delta \right),
\end{aligned}
\tag{33}
$$

where

$$\Gamma^\alpha{}_{\beta\gamma} \equiv \tfrac{1}{2} g^{\alpha\delta} \left(\partial_\beta \, g_{\delta\gamma} + \partial_\gamma \, g_{\delta\beta} - \partial_\delta \, g_{\beta\gamma} \right) = \Delta^{-1} \, X_\alpha \, \mu_\alpha \left(\delta_{\beta\gamma} + \hat{\mu}_\beta \, \hat{\mu}_\gamma \right). \tag{34}$$

Note that ∂_α means $\partial / \partial \mu_\alpha$, and that $\partial^\alpha \equiv g^{\alpha\beta} \, \partial_\beta$.

We calculate the curvature using the expressions

$$
\begin{aligned}
\hat{R}^M{}_{NPQ} &= \partial_P \, \hat{\Gamma}^M{}_{NQ} - \partial_Q \, \hat{\Gamma}^M{}_{NP} + \hat{\Gamma}^M{}_{PR} \, \hat{\Gamma}^R{}_{QN} - \hat{\Gamma}^M{}_{QR} \, \hat{\Gamma}^R{}_{PN} \,, \\
\hat{R}_{NQ} &\equiv \hat{R}^M{}_{NMQ} = \partial_M \, \hat{\Gamma}^M{}_{NQ} - \partial_Q \, \hat{\Gamma}^M{}_{NM} + \hat{\Gamma}^M{}_{MR} \, \hat{\Gamma}^R{}_{QN} - \hat{\Gamma}^M{}_{QR} \, \hat{\Gamma}^R{}_{MN} \,.
\end{aligned}
\tag{35}
$$

After some calculation, we find that

$$
\begin{aligned}
\hat{R}_{\mu\nu} &= R_{\mu\nu} - \tfrac{1}{4} X_i^{-2}\, \partial_\mu X_i\, \partial_\nu X_i + \tfrac{1}{2}\Delta^{-1}\, X_i^{-1}\, \mu_i^2\, \partial_\mu X_i\, \partial_\nu X_i - \tfrac{1}{2}\Delta^{-2}\, \partial_\mu\Delta\, \partial_\nu\Delta \\
&\quad + \tfrac{1}{2}a\,(\Delta^{-2}\, \partial_\lambda\Delta\, \partial^\lambda\Delta - \Delta^{-1}\Box\,\Delta)\, g_{\mu\nu} \\
&\quad - a\left[\sum_i X_i^2 - \Delta^{-1}\, X_i^2\, \mu_i^2 \sum_j X_j - 2\Delta^{-1}\, X_i^3\, \mu_i^2 + 2\Delta^{-2}\,(X_i^2\, \mu_i^2)^2\right] g_{\mu\nu}, \quad (36)
\end{aligned}
$$

$$
\begin{aligned}
\hat{R}_{\alpha\beta} &= R_{\alpha\beta} + \tfrac{1}{2}b\, g_{\alpha\beta}\, \Delta^{-2}\Box\,\Delta - \tfrac{1}{2}b\, g_{\alpha\beta}\, \Delta^{-3}\, \partial_\lambda\Delta\, \partial^\lambda\Delta - \tfrac{1}{2}\Delta^{-1}\Box\, g_{\alpha\beta} \\
&\quad + \tfrac{1}{2}\Delta^{-1}\, g^{\gamma\delta}\, \partial_\lambda g_{\alpha\gamma}\, \partial^\lambda g_{\beta\delta} - \tfrac{1}{4}\Delta^{-2}\, \partial_\alpha\Delta\, \partial_\beta\Delta - \tfrac{1}{2}\Delta^{-1}\, \nabla_\alpha\partial_\beta\,\Delta \\
&\quad - \tfrac{1}{4}b\, g_{\alpha\beta}\, \Delta^{-2}\, \partial_\gamma\Delta\, \partial^\gamma\Delta + \tfrac{1}{2}b\, g_{\alpha\beta}\, \Delta^{-1}\, \nabla_\gamma\partial^\gamma\,\Delta, \quad (37)
\end{aligned}
$$

$$
\hat{R}_{\alpha\mu} = -\tfrac{1}{2}\Delta^{-2}\, U\,(X_\alpha^{-1}\, \partial_\mu X_\alpha - X_0^{-1}\, \partial_\mu X_0)\, \mu_\alpha. \quad (38)
$$

Note that in these expressions $\Box$ means the Laplacian in the lower-dimensional spacetime, ∇_α denotes the covariant derivative with respect to the internal metric $g_{\alpha\beta}$, with its affine connection $\Gamma^\gamma{}_{\alpha\beta}$, and $R_{\alpha\beta}$ is the Ricci tensor calculated in this connection.

Some useful *lemmata* which we used are

$$
\begin{aligned}
\partial^\alpha\Delta\, \partial_\alpha\Delta &= 4 X_i^3\, \mu_i^2 - 4\Delta^{-1}\,(X_i^2\, \mu_i^2)^2, \\
\Gamma^\alpha{}_{\alpha\beta} &= \tfrac{1}{2}\Delta^{-1}\, \partial_\beta\Delta + \frac{1}{\mu_0^2}\,\mu_\beta, \\
\nabla_\alpha\partial^\alpha\Delta &= 2\sum_i X_i^2 - 2\Delta^{-1}\, X_i^2\, \mu_i^2 \sum_j X_j + 4\Delta^{-2}\,(X_i^2\, \mu_i^2)^2 \\
&\quad - 4\Delta^{-1}\, X_i^3\, \mu_i^2 + \tfrac{1}{2}\Delta^{-1}\, \partial^a\Delta\, \partial_\alpha\Delta, \\
R_{\alpha\beta} &= \Delta^{-1}\, \bar{g}_{\alpha\beta} \sum_\gamma X_\gamma - \Delta^{-2}\,(X_i^2\, \mu_i^2)\, \bar{g}_{\alpha\beta} \quad\quad\quad (39) \\
&\quad + \Delta^{-2}\,(X_\alpha - X_0)(X_\beta - X_0)\, \mu_\alpha\, \mu_\beta - \Delta^{-1}\,(X_\alpha - X_0)\, \delta_{\alpha\beta}, \\
\Box\, g_{\alpha\beta} &= X_\alpha^{-3}\, \partial_\lambda X_\alpha\, \partial^\lambda X^\alpha\, \delta_{\alpha\beta} + X_0^{-3}\, \partial_\lambda X_0\, \partial^\lambda X_0\, \hat{\mu}_\alpha\, \hat{\mu}_\beta \\
&\quad - 4(X_\alpha\, \delta_{\alpha\beta} + X_0\, \hat{\mu}_\alpha\, \hat{\mu}_\beta) + 2\bar{g}_{\alpha\beta} \sum_j X_j + \frac{4}{N}\, V\, g_{\alpha\beta}, \\
g^{\gamma\delta}\, \partial_\lambda g_{\alpha\gamma}\, \partial^\lambda g_{\beta\delta} &= X_\alpha^{-3}\, \partial_\lambda X_\alpha\, \partial^\lambda X_\alpha\, \delta_{\alpha\beta} + X_0^{-3}\, \partial_\lambda X_0\, \partial^\lambda X_0\, \hat{\mu}_\alpha\, \hat{\mu}_\beta \\
&\quad - \Delta^{-1}\,(X_\alpha^{-1}\, \partial_\lambda X_\alpha - X_0^{-1}\, \partial_\lambda X_0)(X_\beta^{-1}\, \partial^\lambda X_\beta - X_0^{-1}\, \partial^\lambda X_0)\, \mu_\alpha\, \mu_\beta.
\end{aligned}
$$

The quantities $\hat{\mu}_\alpha$ are defined by $\hat{\mu}_\alpha \equiv \mu_\alpha/\mu_0$, and the metric $\bar{g}_{\alpha\beta}$ is defined by

$$
\bar{g}_{\alpha\beta} \equiv \delta_{\alpha\beta} + \hat{\mu}_\alpha\, \hat{\mu}_\beta. \quad (40)
$$

It is evident from (17) that $\bar{g}_{\alpha\beta}$ is the metric on the unit round $(N-1)$-sphere, corresponding to setting all the $X_i = 1$.

3.2.2 The Consistency of the Einstein Equation

With the results for the Ricci tensor from the previous section, we can now verify that all components of the higher-dimensional Einstein equation are indeed consistently satisfied.

The higher-dimensional Einstein equation is

$$\hat{R}_{MN} = \hat{S}_{MN} \,, \tag{41}$$

where

$$\hat{S}_{MN} = \frac{1}{2\,(D-1)!} \left[\hat{F}^2_{MN} - \frac{D-3}{D\,(D-1)} \, \hat{F}^2 \, \hat{g}_{MN} \right]. \tag{42}$$

The non-zero components of $\hat{F}_{M_1 \cdots M_D}$ are given in (22). After some algebra, we find that

$$
\begin{aligned}
\hat{F}^2 &= -D! \, \Delta^{-Da} \, (U^2 + \Delta \, X_i^{-1} \, \mu_i^2 \, \partial_\lambda X_i \, \partial^\lambda X_i - \partial_\lambda \Delta \, \partial^\lambda \Delta)\,, \\
\hat{F}^2_{\mu\nu} &= (D-1)! \, \Delta^{-2} \Big[\Delta \, X_i^{-1} \, \mu_i^2 \, \partial_\mu X_i \, \partial_\nu X_i - \partial_\mu \Delta \, \partial_\nu \Delta \\
&\qquad\qquad - (\Delta \, X_i^{-1} \, \mu_i^2 \, \partial_\lambda X_i \, \partial^\lambda X_i - \partial_\lambda \Delta \, \partial^\lambda \Delta) \, g_{\mu\nu} - U^2 \, g_{\mu\nu} \Big]\,, \\
\hat{F}^2_{\alpha\beta} &= -(D-1)! \, \Delta^{-2} \, (X_\alpha^{-1} \, \partial_\lambda X_\alpha - X_0^{-1} \, \partial_\lambda X_0)(X_\beta^{-1} \, \partial^\lambda X_\beta - X_0^{-1} \, \partial^\lambda X_0) \, \mu_\alpha \, \mu_\beta \,,
\end{aligned}
\tag{43}
$$

where, as usual, U is given by

$$U = 2 X_i^2 \, \mu_i^2 - \Delta \sum_i X_i \,. \tag{44}$$

Thus we find that $\hat{S}_{MN}$ is given by

$$
\begin{aligned}
\hat{S}_{\mu\nu} &= \tfrac{1}{2} \Delta^{-1} \, X_i^{-1} \, \mu_i^2 \, \partial_\mu X_i \, \partial_\nu X_i - \tfrac{1}{2} \Delta^{-2} \, \partial_\mu \Delta \, \partial_\nu \Delta \\
&\qquad - \frac{1}{D-1} \Delta^{-2} \, (U^2 - \partial_\lambda \Delta \, \partial^\lambda \Delta + \Delta \, X_i^{-1} \, \mu_i^2 \, \partial_\lambda X_i \, \partial^\lambda X_i) \, g_{\mu\nu} \,, \\
\hat{S}_{\alpha\beta} &= \tfrac{1}{2} b \, \Delta^{-3} \, U^2 \, g_{\alpha\beta} + \tfrac{1}{2} b \, \Delta^{-2} \, g_{\alpha\beta} \, X_i^{-1} \, \mu_i^2 \, \partial_\lambda X_i \, \partial^\lambda X_i - \tfrac{1}{2} b \, \Delta^{-3} \, \partial_\lambda \Delta \, \partial^\lambda \Delta \, g_{\alpha\beta} \\
&\qquad - \tfrac{1}{2} \Delta^{-2} \, (X_\alpha^{-1} \, \partial_\lambda X_\alpha - X_0^{-1} \, \partial_\lambda X_0)(X_\beta^{-1} \, \partial^\lambda X_\beta - X_0^{-1} \, \partial^\lambda X_0) \, \mu_\alpha \, \mu_\beta \,, \\
\hat{S}_{\alpha\mu} &= -\tfrac{1}{2} \Delta^{-2} \, U \, (X_\alpha^{-1} \, \partial_\mu X_\alpha - X_0^{-1} \, \partial_\mu X_0) \, \mu_\alpha \,.
\end{aligned}
$$

$$ \tag{45} $$
$$ \tag{46} $$
$$ \tag{47} $$

To verify that the components $\hat{R}_{\mu\nu} = \hat{S}_{\mu\nu}$ of the higher-dimensional Einstein equation indeed imply the lower-dimensional Einstein equation (11), we simply need to substitute the above results into (41). It is also necessary to use the scalar equations of motion in (10), from which we can deduce that

$$\Box \, \Delta = X_i^{-1} \, \mu_i^2 \, \partial_\lambda X_i \, \partial^\lambda X_i + 4 X_i^3 \, \mu_i^2 - 2 X_i^2 \, \mu_i^2 \sum_j X_j - \frac{4}{N} \, \Delta \, V \,. \tag{48}$$

Putting all the results together, we find that indeed all the μ_i dependence cancels out in the $\hat{R}_{\mu\nu} = \hat{S}_{\mu\nu}$ equation, and we correctly reproduce the lower-dimensional Einstein equation in (11).

After some algebra, using the *lemmata* given previously, we find that the components $\hat{R}_{\alpha\beta}$ of the Ricci tensor of the higher-dimensional metric are simply given by

$$
\begin{aligned}
\hat{R}_{\alpha\beta} &= \tfrac{1}{2} b \, \Delta^{-3} U^2 \, g_{\alpha\beta} + \tfrac{1}{2}\Delta^{-2} \, g_{\alpha\beta} \, X_i^{-1} \, \mu_i^2 \, \partial_\lambda X_i \, \partial^\lambda X_i - \tfrac{1}{2} b \, \Delta^{-3} \, \partial_\lambda \Delta \, \partial^\lambda \Delta \\
&\quad - \tfrac{1}{2}\Delta^{-1} \, (X_\alpha^{-1} \, \partial_\lambda X_\alpha - X_0^{-1} \, \partial_\lambda X_0)(X_\beta^{-1} \, \partial^\lambda X_\beta - X_0^{-1} \, \partial^\lambda X_0) \, \mu_\alpha \, \mu_\beta \,.
\end{aligned}
\tag{49}
$$

Note that we have made use of the equations of motion for the X_i fields in simplifying this expression. It is now straightforward to see that this is exactly equal to the expression for $\hat{S}_{\alpha\beta}$ obtained in (46). Finally, the components $\hat{S}_{\alpha\mu}$ given in (47) agree precisely with the corresponding components $\hat{R}_{\alpha\mu}$ found in (38). Thus the consistency of the reduction Ansatz is completely verified.

4 Scalar Potentials in $D = 3$

In the previous sections, we proved the consistency of the embedding of the diagonal symmetric potentials in the relevant higher dimensions. The number of scalars N and the (lower) dimension D are related by (1). As was shown in [9], the various D-dimensional multi-charge extremal AdS domain walls supported by these scalars can be oxidised back to solutions of eleven-dimensional supergravity ($D = 4$ and $D = 7$) or type IIB supergravity $D = 5$). These higher-dimensional solutions correspond to ellipsoidal continuous distributions of M5-branes, M2-branes and D3-branes respectively [11, 12, 13, 9, 14, 15].

For general values of D the relation (1) would imply a non-integral value for N, and no consistent embedding exists. The relation becomes singular for the case $D = 3$. Thus contrary to what one might have hoped, the pattern of consistent embeddings does not seem to extend to an S^3 reduction from $D = 6$ to $D = 3$. Indeed, it is straightforward to show that the ellipsoidal continuous distributions of dyonic strings that exist in $D = 6$ do not lend themselves to consistent reductions to $D = 3$.

In this section, we discuss an alternative reduction to a gauged $D = 3$ supergravity, in which there is a massive scalar field. The three-dimensional bosonic Lagrangian is given by

$$
e^{-1} \mathcal{L}_3 = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}g^2 \left(\frac{1}{a_1^2} e^{a_1\phi} - \frac{1}{a_1 a_2} e^{a_2\phi} \right),
\tag{50}
$$

where $a_1^2 = 4/k + 4$ and $a_2 = 4/a_1$. The integer k can take the values 1, 2, or 3. The values $k = 2$ and $k = 3$ correspond to the S^3 reduction of $D = 6$ simple (chiral) supergravity and

the S^2 reduction of $D = 5$ simple supergravity respectively, and ϕ is the associated massive breathing mode [16].

The case of $k = 1$ corresponds to the S^1 Scherk-Schwarz reduction of the Freedman-Schwarz model. To show this, we begin from the Lagrangian for the gravity plus scalar sector of the $D = 4$ Freedman-Schwarz model [17], which can be obtained as a singular limit [8] of the $N = 4$, $D = 4$, $SO(4)$ gauged supergravity [18]:

$$\hat{e}^{-1}\mathcal{L}_4 = \hat{R} - \tfrac{1}{2}(\partial\hat{\phi})^2 - \tfrac{1}{2}(\partial\chi)^2\, e^{2\hat{\phi}} + \frac{1}{2}g^2 e^{\hat{\phi}} \,. \tag{51}$$

Dimensionally reducing this theory on a coordinate z, where the axion χ is allowed to take the generalised Scherk-Schwarz form $\chi = m\, z$, we obtain the three-dimensional scalar Lagrangian

$$e^{-1}\mathcal{L}_3 = R - \tfrac{1}{2}(\partial\hat{\phi})^2 - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}m^2\, e^{2(\hat{\phi}+\varphi)} + \tfrac{1}{2}g^2\, e^{\hat{\phi}+\varphi} \,. \tag{52}$$

Since the original dilaton $\hat{\phi}$ and the dilaton φ coming from the dimensional reduction occur everywhere in the same combination, we see that it is consistent to truncate out the combination $\hat{\phi} - \varphi$. Making the redefinition $\phi \equiv (\hat{\phi} + \varphi)/\sqrt{2}$, the Lagrangian (52) reduces to (50) with $k = 1$. The three Lagrangians in (50) all give rise to supersymmetric domain-wall solutions in $D = 3$ [16, 19].

5   Conclusion

In this paper, we have provided a complete proof of the consistency of the Kaluza-Klein reduction Ansätze that were presented in [9], which describe the embedding of certain N-scalar truncations of the maximal gauged supergravities in $D = 4$, 7 and 5, *via* spherical reductions on S^7, S^4 and S^5 respectively. The N scalars, with $N = 8$, 5 and 6 respectively, correspond to the diagonal elements in the $SL(N, R)/SO(N)$ submanifolds of the full scalar manifolds in the corresponding maximal supergravities. (Actually, there are really only $N - 1$ independent scalars in these truncations, on account of a unit-determinant condition on the scalars in the coset.) Our proof included a complete verification of the consistency of the reduction of the higher-dimensional Einstein equation, which is usually the most calculationally difficult part of the procedure.

For $D = 7$, our results are consistent with the full Kaluza-Klein S^4 reduction that was recently obtained explicitly in [2, 3]. For $D = 4$, they are compatible with the implicit proof of the consistency of the complete S^7 reduction, presented in [1]. Furthermore, our results provide a complete proof of the validity of the explicit expressions presented in [9] for the

Ansätze for the eleven-dimensional fields, which, especially in the case of the 4-form field strength, are not straightforward to extract from the results presented in [1].

Finally, in $D = 5$ our results provide further evidence for the conjectured consistency of the S^5 reduction of type IIB supergravity to give maximal $SO(6)$ gauged supergravity in $D = 5$.

We also considered the special case of scalar theories in $D = 3$ that arise from dimensional reduction. This dimension lies outside the set of cases covered by the previous discussion, on account of a degeneration in the formula (1) relating the dimension to the number of scalar fields. Instead, we described the set of three theories (50) arising as the scalar sectors of sphere reductions from $D = 6$, $D = 5$ and $D = 4$. In the case of $D = 4$, we showed how the single-scalar Lagrangian (50) arises from a Scherk-Schwarz S^1 reduction of the $D = 4$ Freedman-Schwarz model, accompanied by a further consistent truncation of one combination of the two resulting dilatonic scalar fields.

Acknowledgements

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